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QUARTET FIXED POINTS FOR MAPS ON A MULTIPLICATIVE METRIC SPACE

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***Abstract*:** In this paper we introduce the notion of a quartet fixed point for maps on a multiplicative metric space. We prove the existence and uniqueness of a quartet fixed point for such maps. Two supporting examples are also given.

***Keywords* :** Multiplicative metric space, multiplicative contraction, Tripled fixed point, Quartet fixed point, Mixed monotone property, Common fixed point.

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1. INTRODUCTION AND PRELIMINARIES:

In 1906, F.Frechet first introduced metric space. Till now, so many spaces are generalized from metric space. Among the generalized metric space, multiplicative metric space is one. The definition of multiplicative metric space was given by Michael Grossman and Robert Katzin 1967-1970.

In 2012, Ozavsar and Cevikel [7] introduced the concept of multiplicative contraction mapping and proved some fixed point theorems for this type of mappings. In 1987, Guo and Lakshmikantham [2] first given the definition of coupled fixed point. Later, Bhaskar and Lakshmikantham [10] proved a new fixed point theorem for a mixed monotone mapping in a metric space by using a weak contractivity type assumption with applications.

In, 2011, V. Berinde and M. Borcut [12] introduced the concept of triple fixed point for nonlinear mappings in partially ordered complete metric spaces and obtained existence and uniqueness theorems for contractive type mappings.

In this paper we first prove a tripled fixed point theorem and obtain the result of L. Shanjit and Y. Rohen [6] as a corollary. Further we introduce the concept of quartet fixed point and prove fixed point theorem in a partially ordered multiplicative metric space, which is an extension of the result 2.1 of L. Shanjit and Y. Rohen [6]. Two supporting examples are provided.

Definition 1.1. (A.E.Bashirov, E.M. Kurpinara., Ozyapici [1]). Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to \mathrm{R}^+$ satisfying the following conditions:

(i) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$, if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, y) \leq d(x, z).d(z, y)$ for all $x, y, z \in X$. (Multiplicative triangle inequality)

Also (X, d) is called a multiplicative metric space.

Note that R^+ is a multiplicative metric space with respect to the multiplication.

Example 1.2. (M.Ozavser, A.C. Cevikel [7]). Let $d^* : (\mathrm{R}^+)^n \times (\mathrm{R}^+)^n \to \mathrm{R}^+$ be defined as follows

$$d^*(x, y) = \left|\frac{x_1}{y_1}\right|^* . \left|\frac{x_2}{y_2}\right|^* ...... \left|\frac{x_n}{y_n}\right|^* .$$

where $x = (x_1, x_2, ...x_n), y = (y_1, y_2, ...y_n) \in \mathrm{R}^+$ and $|\ |^* : \mathrm{R}^+ \to \mathrm{R}^+$ is

$$d^* = \begin{cases} a & \text{if } a \geq 1 \\ \frac{1}{a} & \text{if } a \leq 1 \end{cases}$$ Then $((R^+)^n, d^*)$ is a multiplicative metric space.

Example 1.3. (M.Ozavser, A.C. Cevikel [7]). Let $a > 1$ be fixed real number. Then $d_a : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$d_a(w,z) = a^{\sum_{i=1}^{n}|w_i - z_i|}$ where $w = (w_1, w_2, ...w_n), z = (z_1, z_2, ...z_n) \in \mathbb{R}^n$.

Obviously, $(\mathbb{R}^n, d_a)$ is a multiplicative metric space. We can also extended multiplicative metric C^n by the following definition: $d_a(w,z) = a^{\sum_{i=1}^{n}|w_i - z_i|}$ where $w = (w_1, w_2, ...w_n), z = (z_1, z_2, ...z_n) \in C^n$.

Definition 1.4. (M.Ozavser, A.C.Cevikel. [7]). (Multiplicative convergence). Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\varepsilon(x) = \{y / d(x,y) < \varepsilon\}, \varepsilon > 1$ there exists a natural number N such that for $n \geq N$, $x_n \in B_\varepsilon(x)$, the sequence $\{x_n\}$ is said to be multiplicative converging to x, denoted by $x_n \to x \ (n \to \infty)$.

Definition 1.5. (M.Ozavser, A.C. Cevikel [7]). Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called a multiplicative Cauchy sequence if, for each $\varepsilon > 1$, there exists $N \in \mathbf{N}$ such that $d(x_n, x_m) < \varepsilon$, for all $m, n \geq N$.

Definition 1.6. (M.Ozavser, A.C. Cevikel [7]). Let (X, d) be a multiplicative metric space. We call (X, d) is complete if every multiplicative Cauchy sequence in X is multiplicative convergent to $x \in X$

Definition 1.7. (M.Ozavser, A.C. Cevikel [7]). Let (X, d) be a multiplicative metric space. A mapping $f : X \to X$ is called a multiplicative contraction if there exists a real constant $\lambda \in [0,1)$ such that $d(fx, fy) \leq d(x,y)^\lambda$ for all $x, y \in X$.

Definition 1.8. (M.Ozavser, A.C. Cevikel [7]). Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces and $f : X \to Y$ be a function. If for every $\varepsilon > 1$, there exists $\delta > 1$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$, then we call f multiplicative continuous at $x \in X$.

Lemma 1.9. (M.Ozavser, A.C. Cevikel [7]). Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $x_n \to x \ (n \to \infty)$ if and only if $d(x_n, x) \to 1 \ (n \to \infty)$.

Lemma 1.10. (M.Ozavser, A.C. Cevikel [7]). Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if
$d(x_n, x_m) \to 1 \ (m, n \to \infty)$.

Definition 1.11. (T. Gnana Bhaskar, V. Lakshmikantham [10]). Let $(X, \preceq)$ be a partially ordered set and $S : X \times X \to X$. The mapping S is said to have the mixed monotone property if S is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow S(x_1, y) \preceq S(x_2, y) \text{ and } y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow S(x, y_1) \pm S(x, y_2)$$

Definition 1.12. (T. Gnana Bhaskar, V. Lakshmikantham [10]). Let $(X, \preceq)$ be a partially ordered set, an element (x, y) is called a coupled fixed point of the mapping $S : X \times X \times X \to X$ if
$S(x, y) = x$, $S(y, x) = y$.

Definition 1.13. (V. Berinde, M. Borcut [12]). Let $(X, \preceq)$ be a partially ordered set and $G : X \times X \times X \to X$. The mapping G is said to have the mixed monotone property if for any
$x, y \in X$

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow G(x_1, y, z) \preceq G(x_2, y, z)$$
$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow G(x, y_1, z) \pm G(x, y_2, z)$$
$$z_1, z_2 \in X, z_1 \preceq z_2 \Rightarrow G(x, y, z_1) \preceq G(x, y, z_2).$$

Definition 1.14. (V. Berinde, M. Borcut [12]). $G : X \times X \times X \to X$. An element (x, y, z) is called a tripled fixed point of G if $G(x, y, z) = x$, $G(y, x, y) = y$ and $G(z, y, x) = z$

Theorem 1.15. (V. Berinde, M. Borcut [12]). Let $(X,\preceq,d)$ be a partially ordered set and suppose d is a multiplicative metric on X such that (X,d) is a complete multiplicative metric space. Suppose $F: X\times X\times X \to X$ is such that F has mixed monotone property and there exist $j,r,l \geq 0$ with $j+r+l<1$ such that $d(F(x,y,z),F(u,v,w)) \leq$ $jd(x,u)+rd(y,v)+ld(z,w)$ for any $x,y,z\in X$ for which $x \preceq u$, $v\preceq y$ and $z \preceq w.$ Suppose either F is continuous, or X has the following property:

1. if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x \ \forall n.$
2. if a non-increasing sequence $y_n \to y$, then $y_n \pm y \ \forall n.$

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0,y_0,z_0)$, $y_0 \pm F(y_0,x_0,y_0)$ and $z_0 \preceq F(z_0,y_0,x_0)$, then there exist $x,y,z\in X$ such that $F(x,y,z)=x,$ $F(y,x,y)=y$ and $F(z,y,x)=z$ that is, F has a tripled fixed point.

Definition 1.16. (H. Aydi, E. Karapinar [3]). Let (X,d) be a multiplicative metric space. A mapping $T: X\to X$ is said to be **ICS** if T is injective, continuous and has the property : for every sequence $\{x_n\}$ in X, if $\{Tx_n\}$ is convergent then $\{x_n\}$ is also convergent.

Let Φ be the set of all functions $\varphi:[0,\infty)\to[0,\infty)$ such that

(1) φ is non-decreasing,

(2) $\varphi(t)<t$ for all $t>0$,

(3) $\lim_{r\to t^+}\varphi(r)<t \ \ for\ all\ t>0.$

Recently L.Shanjiti and Y. Rohen [6] proved the following tripled fixed point theorem in multiplicative metric spces.

Theorem 1.17. (L. Shanjit and Y. Rohen [6]). Let (X,d) be a partially ordered set and suppose there is a multiplicative metric d on X such that (X,d) is a complete multiplicative metric space. suppose $T: X\to X$ is an **ICS** mapping and $F: X\times X\times X \to X$ is such that F has mixed monotone property. Assume that there exists $\varphi\in\Phi$ such that

$d(TF(x,y,z),TF(u,v,w)) \leq \varphi(max\{d(Tx,Tu),d(Ty,Tv),d(Tz,Tw)\})$

for any $x,y,z\in X$ for which $x\preceq u,$ $y\pm v,$ and $z\preceq w$. Suppose either

(a) F is continuous, or

(b) F has the following property:

(1) If non-decreasing sequence $x_n\to x$ (respectively, $z_n \to z$), then $x_n\preceq x$ (respectively $z_n\preceq z$) $\forall n.$

(2) If non-increasing sequence $y_n\to y$, then $y_n \pm y \ \forall n.$

If there exists $x_0,y_0,z_0\in X$ such that $x_0\preceq F(x_0,y_0,z_0)$, $y_0 \pm F(y_0,x_0,y_0)$ and $z_0\preceq F(z_0,y_0,x_0)$ then there exist $x,y,z\in X$ such that $F(x,y,z)=x,$ $F(y,x,y)=y$ and $F(z,y,x)=z$ that is, F has a tripled fixed point.

Suppose that for all (x,y,z), $(u,v,r) \in X\times X\times X$, there exists $(a,b,c)\in X\times X\times X$ such that $(F(a,b,c),F(b,a,b),F(c,b,a))$ is comparable to $(F(x,y,z),F(y,x,y),F(z,y,x))$ and $(F(u,v,r),F(v,u,v),F(r,v,u))$. Then F has a unique tripled point (x,y,z).

2. MAIN RESULT.

In this section we first introduce the notion of quartet fixed point for a function of four variables on a partially ordered multiplicative metric space.

Defini tion 2.1. Let $(X,\preceq,d)$ be a partially ordered multiplicative metric space and $G: X\times X\times X\times X\to X$. An element (x,y,z,s) is called a quartet fixed point of G if $G(x,y,z,s)=x$, $G(y,z,s,x)=y$, $G(z,s,x,y)=z$ and $G(s,x,y,z)=s$.

Defini tion 2.2. Let $(X,\preceq)$ be a partially ordered set and $G: X\times X\times X\times X\to X$. The mapping G is said to have the mixed monotone property if G is monotone non-decreasing in its first and third argument and is monotone non-increasing in its second and fourth argument, that is, for any $x,y,z,s\in X$,

$x_1,x_2\in X, x_1\preceq x_2 \Rightarrow G(x_1,y,z,s)\preceq G(x_2,y,z,s)$

$y_1,y_2\in X, y_1\preceq y_2 \Rightarrow G(x,y_1,z,s)\pm G(x,y_2,z,s)$

$z_1,z_2\in X, z_1\preceq z_2 \Rightarrow G(x,y,z_1,s)\preceq G(x,y,z_2,s)$

$s_1,s_2\in X, s_1\preceq s_2 \Rightarrow G(x,y,z,s_1)\pm G(x,y,z,s_2)$.

Notation:

Let Φ be the set of all functions $\varphi:[1,\infty)\to[1,\infty)$ such that

(i). φ is non-decreasing , $\varphi(1)=1,$

(ii) $\varphi(t)<t$ for all $t>1$,

(iii) $\lim_{r\to t^+}\varphi(r)<t$ *for all* $t>1.$

Now we prove quartet fixed point theorems in partially ordered multiplicative metric space, which is a generalization of the result 2.1 of L. Shanjit and Y. Rohen [6]. First we prove a tripled fixed point theorem in partially ordered multiplicative metric space and obtain the result of [6] as a corollary.

Theorem 2.3. Let $(X,\preceq,d)$ be a complete partially ordered multiplicative metric space and $G:X\times X\times X\to X$ be such that G has mixed monotone property.

Assume that there exists $\varphi\in\Phi$ such that

$$d(G(x,y,z),G(u,v,w))\ \le\ \varphi(max\{d(x,u),d(y,v),d(z,w)\}) \qquad (2.3.1)$$

for any $x,y,z\in X$ for which $x\preceq u,\ \ y\pm u$ and $z\preceq w$. Suppose either

(a) G is continuous, or

(b) G has the following property:

(1) If non-decreasing sequence $x_n\to x$ then $x_n\preceq x\ \ \forall n.$

(2) If non-increasing sequence $y_n\to y$, then $y_n\pm y\ \ \forall n.$

If there exists $x_0,y_0,z_0\in X$ such that $x_0\preceq G(x_0,y_0,z_0)$, $y_0\pm G(y_0,x_0,y_0)$ and $z_0\preceq G(z_0,y_0,x_0)$ then there exist $x,y,z\in X$ such that $G(x,y,z)=x,\ \ G(y,x,y)=y$ and $G(z,y,x)=z$ that is, G has a tripled fixed point.

Suppose that for all (x,y,z), $(u,v,w)\ \in X\times X\times X$, there exists $(a,b,c)\ \in X\times X\times X$ such that $(G(a,b,c),G(b,a,b),G(c,b,a))$ is comparable to $(G(x,y,z),G(y,x,y),G(z,y,x))$ and $(G(u,v,w),G(v,u,v),G(w,v,u))$. Then G has a unique tripled point (x,y,z).

Proof: Let $x_0,y_0,z_0\in X$ such that $x_0\preceq G(x_0,y_0,z_0)$, $y_0\preceq G(y_0,x_0,y_0)$ and $z_0\preceq G(z_0,y_0,x_0)$

Set $x_1=G(x_0,y_0,z_0)$, $y_1=G(y_0,x_0,y_0)$ and $z_1=G(z_0,y_0,x_0)$. (2.3.2)

Continuing this process, we can construct sequences $\{x_n\},\{y_n\}$ and $\{z_n\}$ in X such that $x_{n+1}=G(x_n,y_n,z_n)$,

$y_{n+1}=G(y_n,x_n,y_n)$ and $z_{n+1}=G(z_n,y_n,x_n)$. (2.3.3)

Since G has the mixed monotone property, then using mathematical induction,

we have $x_n\preceq x_{n+1}$, $y_{n+1}\pm y_n$ and $z_n\preceq z_{n+1}$. (2.3.4)

Assume for some $n\in\mathbf{N},$

$x_n=x_{n+1}$, $y_{n+1}=y_n$ and $z_n=z_{n+1}$. i.e., $(x_n,y_n,z_n)=(x_{n+1},y_{n+1},z_{n+1})$

Then by (2.3.3), (x_n,y_n,z_n) is a tripled fixed point of G.

From now on, assume that for any $n\in\mathbf{N},\ (x_n,y_n,z_n)\ \ne\ (x_{n+1},y_{n+1},z_{n+1})$

i.e., $x_n\ne x_{n+1}$ or $y_n\ne y_{n+1}$ and $z_n\ne z_{n+1}$. (2.3.5)

Then by (2.3.5), for any $n\in\mathbf{N},$

$$a_{n+1}=max\{d(x_n,x_{n+1}),d(y_n,y_{n+1}),d(z_n,z_{n+1})\}>1.$$

From (2.3.1) and (2.3.3), we have

$$\begin{aligned}d(x_n,x_{n+1})&=d(G(x_{n-1},y_{n-1},z_{n-1}),G(x_n,y_n,z_n))\\&\le\ \varphi(max\{d(x_{n-1},x_n),d(y_{n-1},y_n),d(z_{n-1},z_n)\}) \qquad (2.3.6)\end{aligned}$$

$$\begin{aligned}d(y_{n+1},x_n)&=d(G(y_n,x_n,y_n),G(y_{n-1},x_{n-1},y_{n-1}))\\&\le\ \varphi(max\{d(y_n,y_{n-1}),d(x_n,x_{n-1}),d(y_n,y_{n-1})\})\\&=\varphi(max\{d(y_n,y_{n-1}),d(x_n,x_{n-1})\})\\&\le\ \varphi(max\{d(y_n,y_{n-1}),d(x_n,x_{n-1}),d(z_n,z_{n-1})\}) \qquad (2.37)\end{aligned}$$

$$\begin{aligned}d(z_n,z_{n+1})&=d(G(z_{n-1},y_{n-1},x_{n-1}),G(z_n,y_n,x_n),)\\&\le\ \varphi(max\{,d(z_{n-1},z_n),d(y_{n-1},y_n),d(x_{n-1},x_n)\}) \qquad (2.3.8)\end{aligned}$$

since, $\varphi(t) < t \quad \forall t > 1$, so from (2.3.6) to (2.3.8) we get that

$$1 < a_{n+1} = max\{d(x_n, x_{n+1}), d(y_n, y_{n+1}), d(z_n, z_{n+1})\}$$
$$\leq \varphi(max\{d(x_n, x_{n-1}), d(y_n, y_{n-1}), d(z_n, z_{n-1})\})$$
$$< max\{d(x_{n-1}, x_n), d(y_{n-1}, y_n), d(z_{n-1}, z_n)\} = a_n \quad (2.3.9)$$

It follows that $a_{n+1} < a_n$

Thus a_{n+1} is a positive decreasing sequence. Hence there exists $r \geq 1$ such that a_{n+1} decreases to r.

Suppose $r > 1$, in (2.3.9), we get that $1 < \varphi(a_n) \leq a_n$.

On letting $n \to \infty$

$$1 \leq r \leq \lim_{n\to\infty} \varphi(a_n) = \lim_{a_n \to r^+} \varphi(a_n) < r = \lim_{n\to\infty} a_n. \quad (2.3.10)$$

which is a contradiction. $\therefore r = 1.$

i.e., $\lim_{n\to\infty}(a_{n+1}) = \lim_{n\to\infty} max\{d(x_n, x_{n+1}), d(y_n, y_{n+1}), d(z_n, z_{n+1})\} = 1.$ (2.3.11)

Now we shall prove that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are Cauchy sequences.

Assume the contrary, i.e., $\{x_n\}$, $\{y_n\}$, or $\{z_n\}$ is not Cauchy sequence.

that is $\lim_{n,m\to\infty} d(x_m, x_n) \neq 1$ or $\lim_{n,m\to\infty} d(y_m, y_n) \neq 1$ or $\lim_{n,m\to\infty} d(z_m, z_n) \neq 1$

This means that there exists $\varepsilon > 1$ for which we can find subsequences of integers $\{m_k\}$, and $\{n_k\}$ with $\{n_k\} > \{m_k\} > k$ such that $max\{d(x_{m_k}, x_{n_k}), d(y_{m_k}, y_{n_k}), d(z_{m_k}, z_{n_k})\} \geq \varepsilon$ (2.3.12)

Further, corresponding to $\{m_k\}$ we can choose $\{n_k\}$ in such a way that it is smallest integer with $\{n_k\} > \{m_k\}$ and satisfying (2.3.12)

Then $max\{d(x_{m_k}, x_{n_{k-1}}), d(y_{m_k}, y_{n_{k-1}}), d(z_{m_k}, z_{n_{k-1}})\} \geq \varepsilon$ (2.3.13)

by triangular inequality and (2.3.3), we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_{k-1}}).\{d(x_{n_{k-1}}, x_{n_k})$$
$$< \varepsilon.\{d(x_{n_{k-1}}, x_{n_k})$$

Thus, by (2.3.11), we get that

$$\varepsilon \leq \lim_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq \lim_{k\to\infty} d(x_{m_k}, x_{n_{k-1}}) \leq \varepsilon \quad (2.3.14)$$

Similarly, we have $\varepsilon \leq \lim_{k\to\infty} d(y_{m_k}, y_{n_k}) \leq \lim_{k\to\infty} d(y_{m_k}, y_{n_{k-1}}) \leq \varepsilon$ (2.3.15)

and $\varepsilon \leq \lim_{k\to\infty} d(z_{m_k}, z_{n_k}) \leq \lim_{k\to\infty} d(z_{m_k}, z_{n_{k-1}}) \leq \varepsilon$ (2.3.16)

Again by (2.3.13), we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_{k-1}}).d(x_{m_{k-1}}, x_{n_{k-1}}).d(x_{n_{k-1}}, x_{n_k})$$
$$\leq d(x_{m_k}, x_{m_{k-1}}).d(x_{m_{k-1}}, x_{m_k}).d(x_{m_k}, x_{n_{k-1}}).d(x_{n_{k-1}}, x_{n_k})$$
$$\leq d(x_{m_k}, x_{m_{k-1}}).d(x_{m_{k-1}}, x_{m_k}).\grave{o}.d(x_{n_{k-1}}, x_{n_k})$$

Letting $k \to \infty$ and using (2.3.11), we get

$$\varepsilon \leq \lim_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq 1.1.\varepsilon.1. = \varepsilon. \quad (2.3.17)$$

Similarly, $\varepsilon \leq \lim_{k\to\infty} d(y_{m_k}, y_{n_k}) \leq \varepsilon.$ (2.3.18)

$$\varepsilon \leq \lim_{k\to\infty} d(z_{m_k}, z_{n_k}) \leq \varepsilon. \quad (2.3.19)$$

Now, using (2.3.12) and (2.3.17)-(2.3.19),we have

$$\lim_{k\to\infty}\{d(x_{m_k}, x_{n_k}), d(y_{m_k}, y_{n_k}), d(z_{m_k}, z_{n_k})\} = \varepsilon. \quad (2.3.20)$$

Now using (2.3.1), we obtain

$$d(x_{m_k}, x_{n_k}) = d(G(x_{m_{k-1}}, y_{m_{k-1}}, z_{m_{k-1}}), G(x_{n_{k-1}}, y_{n_{k-1}}, z_{n_{k-1}}))$$
$$\leq \varphi(max\{d(x_{m_{k-1}}, x_{n_{k-1}}), d(y_{m_{k-1}}, y_{n_{k-1}}), d(z_{m_{k-1}}, z_{n_{k-1}})\}) \quad (2.3.21)$$

$$d(y_{m_k}, y_{n_k}) = d(G(y_{m_{k-1}}, x_{m_{k-1}}, y_{m_{k-1}}), G(y_{n_{k-1}}, x_{n_{k-1}}, y_{n_{k-1}}))$$

$$\le \varphi(max\{d(y_{m_{k-1}}, y_{n_{k-1}}), d(x_{m_{k-1}}, x_{n_{k-1}}), d(y_{m_{k-1}}, y_{n_{k-1}})\})$$
$$= \varphi(max\{d(y_{m_{k-1}}, y_{n_{k-1}}), d(x_{m_{k-1}}, x_{n_{k-1}})\})$$
$$\le \varphi(max\{d(x_{m_{k-1}}, x_{n_{k-1}}), d(y_{m_{k-1}}, y_{n_{k-1}}), d(z_{m_{k-1}}, z_{n_{k-1}})\}) \quad (2.3.22)$$

and $d(z_{m_k}, z_{n_k}) = d(G(z_{m_{k-1}}, y_{m_{k-1}}, x_{m_{k-1}}), G(z_{n_{k-1}}, y_{n_{k-1}}, x_{n_{k-1}}))$

$$\le \varphi(max\{d(x_{m_{k-1}}, x_{n_{k-1}}), d(y_{m_{k-1}}, y_{n_{k-1}}), d(z_{m_{k-1}}, z_{n_{k-1}})\}) \quad (2.3.23)$$

From (2.3.21) to (2.3.23) we get that

$$max\{d(x_{m_k}, x_{n_k}), d(y_{m_k}, y_{n_k}), d(z_{m_k}, z_{n_k})\}$$
$$\le \varphi(max\{d(x_{m_{k-1}}, x_{n_{k-1}}), d(y_{m_{k-1}}, y_{n_{k-1}}), d(z_{m_{k-1}}, z_{n_{k-1}})\}) = \varepsilon \quad (2.3.24)$$

Letting $k \to \infty$ in (2.3.24) and having in mind (2.3.20), we get

$1 < \varepsilon \le \lim_{t \to \grave{o}^+} \varphi(t) < \varepsilon$, which is a contradiction.

Thus $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are Cauchy sequences in X.

Since X is complete multiplicative metric space, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are convergent sequences.

So there exist $x, y, z \in X$ such that $x_n \to x$, $y_n \to y$ and $z_n \to z$ (2.3.25)

Suppose now the assumption (a) holds, that is G is continuous. By (2.3.3) and (2.3.25) we obtain

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} G(x_n, y_n, z_n)$$
$$= G(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n,, \lim_{n\to\infty} z_n) = G(x, y, z)$$

Similarly $y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} G(y_n, x_n, y_n)$

$$= G(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = G(y, x, y)$$

and $z = \lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} G(z_n, y_n, x_n)$

$$= G(\lim_{n\to\infty} z_n, \lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = G(z, y, x)$$

$\therefore (x, y, z)$ is a tripled fixed point of G.

Suppose the assumption (b) holds,

i.e., $\{x_n\}$, $\{z_n\}$ are non-decreasing with $x_n \to x$, $z_n \to z$ and also $\{y_n\}$ is non-increasing with $y_n \to y$, then we have $x_n \preceq x$, $z_n \preceq z$ and $y_n \pm y$.

for all n, , Consider now

$$d(x, G(x, y, z)) \le d(x, x_{n+1}).d(x_{n+1}, G(x, y, z))$$
$$= d(x, x_{n+1}).d(G(x_n, y_n, z_n), G(x, y, z))$$
$$\le d(x, x_{n+1}) . \varphi(max\{d(x_n, x), d(y_n, y), d(z_n, z)\}) \quad (2.3.26)$$

For infinitely many n, $\varphi(a_n) < a_n$

$$1 \le \lim_{n\to\infty} d(x, G(x, y, z)) \le \lim_{n\to\infty}[d(x, x_{n+1}).max\{d(x_n, x), d(y_n, y), d(z_n, z)\})] \quad \to 1$$

$$1 \le \lim_{n\to\infty} d(x, G(x, y, z)) \le 1$$

$$d(x, G(x, y, z)) = 1 \qquad \therefore x = G(x, y, z)$$

Similarly we can show that $y = G(y, x, y)$ and $z = G(z, y, x)$

Hence (x, y, z) is a tripled fixed point of G.

<u>Uniqueness of tripled fixed point:</u>

Let (x, y, z) and (u, v, w) are two tripled fixed points of G.

Suppose $(x, y, z) \preceq (u, v, w)$ i.e., $x \preceq u$, $y \pm v$ and $z \preceq w$.

$$d(x, u) = d(G(x, y, z), G(u, v, w)) \le \varphi(max\{d(x, u), d(y, v), d(z, w)\})$$
$$= \varphi(d(x, u) < d(x, u) \qquad (\text{ if } d(x, u) \text{ is maximum})$$

$\therefore d(x, u) < d(x, u)$, a contradiction, if $d(x, u) > 1$.

$\therefore d(x,u)=1, d(y,v)=1, d(z,w)=1$
$\therefore x=u, y=v, z=w$
$\therefore (x,y,z)=(u,v,w)$.

By our assumption, there exists $(a,b,c) \in X\times X\times X$ such that $(G(a,b,c),G(b,a,b),G(c,b,a))$ is comparable to $(G(x,y,z),G(y,x,y),G(z,y,x))$ and $(G(u,v,w),G(v,u,v),G(w,v,u))$.

Define sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ such that $a_0=a$, $b_0=b$ and $c_0=c$

and for any $n\geq 1$,

$a_n=G(a_{n-1},b_{n-1},c_{n-1})$, $b_n=G(b_{n-1},a_{n-1},b_{n-1})$ and $c_n=G(c_{n-1},b_{n-1},a_{n-1})$ $\quad\forall n$ (2.3.27)

for all n.

Further set $x_0=x,\ y_0=y,\ z_0=z$ and $u_0=u,\ v_0=v,\ w_0=w,$

and in the same way we define the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$.

Then it follows that $G(x,y,z)=x_n$, $G(y,x,y)=y_n$, $G(z,y,x)=z_n$

$$G(u,v,r)=u_n,\ G(v,u,v)=v_n,\ G(r,v,u)=r_n\ \ \forall n\ \geq 1. \quad (2.3.28)$$

since $(G(x,y,z),G(y,x,y),G(z,y,x))=(x_1,y_1,z_1)=(x,y,z)$ is comparable to $(G(a,b,c),G(b,a,b),G(c,b,a))=(a_1,b_1,c_1)$

Then it is enough to show that $(x,y,z)\ \pm\ (a_1,b_1,c_1)$

Recursively, we get that $(x,y,z)\pm(a_n,b_n,c_n)\quad \forall n$ (2.3.29)

By (2.3.29) and (2.3.1), we have

$$d(x,a_{n+1})=d(G(x,y,z),G(a_n,b_n,c_n))$$
$$\leq \varphi(max\{d(x,a_n),d(y,b_n),d(z,c_n)\}) \quad (2.3.30)$$
$$d(b_{n+1},y)=d(G(b_n,a_n,b_n),G(y,x,y))$$
$$\leq \varphi(max\{d(b_n,y),d(a_n,x),d(b_n,y)\})$$
$$=\varphi(max\{d(b_n,y),d(a_n,x)\})$$
$$\leq \varphi(max\{d(b_n,y),d(a_n,x),d(c_n,z)\}) \quad (2.3.31)$$

and $d(z,c_{n+1})=d(G(z,y,x),G(c_n,b_n,a_n))$

$$\leq \varphi(max\{d(z,c_n),d(y,b_n),d(x,a_n)\}) \quad (2.3.32)$$

It follows from (2.3.30)-(2.3.32) that

$$max\{d(x,a_{n+1}),d(y,b_{n+1}),d(z,c_{n+1})\}\ \leq\ \varphi(max\{d(x,a_n),d(y,b_n),d(z,c_n)\})$$

there fore, for each $n\geq 1$,

$$(max\{d(z,c_n),d(y,b_n),d(x,a_n)\})\ \leq\ \varphi(max\{d(z,c_{n-1}),d(y,b_{n-1}),d(x,a_{n-1})\})$$
$$\leq\ \varphi^n(max\{d(x,a_0),d(y,b_0),d(z,c_0)\}) \quad (2.3.33)$$

We know that $\varphi(t)<t$ and $\lim_{r\to t^+}\varphi(r)\ <t \Rightarrow \lim_{n\to\infty}\varphi^n(t)=1$

for each $t>1$, thus, from(2.3.33)

$$\lim_{n\to\infty}\max\{d(x,a_n),d(y,b_n),d(z,c_n)\}=1$$

i.e., $\lim_{n\to\infty}d(x,a_n)=1$, $\lim_{n\to\infty}d(y,b_n)=1$, and $\lim_{n\to\infty}d(z,c_n)=1$ (2.3.34)

Analogously, we show that

$$\lim_{n\to\infty}d(u,a_n)=1,\ \lim_{n\to\infty}d(v,b_n)=1 \text{ and } \lim_{n\to\infty}d(r,c_n)=1 \quad (2.3.35)$$

Combining (2.3.34) and (2.3.35), it follows that (x,y,z) and (u,v,w) are equal.

$\therefore x=u, y=v,$ and $z=w.$ Hence Uniqueness holds.

Corollary 2.4. Let $(X,\preceq)$ be a partially ordered set and suppose there is a multiplicative metric d on X such that (X,d) is a complete multiplicative metric space. Suppose $T:X\to X$ is an **ICS** mapping and $F:X\times X\times X\to X$ be such that F has mixed monotone property. Assume that there exists $\varphi\in\Phi$ such that $d(TF(x,y,z),TF(u,v,w))\ \leq$ $\varphi(max\{d(Tx,Tu),d(Ty,Tv),d(Tz,Tw)\})$

for any $x, y, z \in X$ for which $x \preceq u$, $y \pm v$, and $z \preceq w$. Suppose either

(a) F is continuous, or

(b) F has the following property:

(1) If non-decreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \preceq x$ (respectively $z_n \preceq z$) $\forall n.$

(2) If non-increasing sequence $y_n \to y$, then $y_n \pm y$ $\forall n.$

If there exists $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \pm F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$ then there exist $x, y, z \in X$ such that $F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, y, x) = z$ that is, F has a tripled fixed point.

Suppose that for all (x, y, z), $(u, v, w) \in X \times X \times X$, there exists $(a, b, c) \in X \times X \times X$ such that $(F(a,b,c), F(b,a,b), F(c,b,a))$ is comparable to $(F(x,y,z), F(y,x,y), F(z,y,x))$ and $(F(u,v,w), F(v,u,v), F(w,v,u))$. Then F has a unique tripled point (x, y, z).

Proof: Put $G = TF$ in theorem 2.3

Now we state and prove quartet fixed point theorem.

Theorem 2.5. Let $(X, \preceq, d)$ be a complete partially ordered multiplicative metric space and $G : X \times X \times X \times X \to X$ be such that G has mixed monotone property.

Assume that there exists $\varphi \in \Phi$ such that

$$d(G(x,y,z,s), G(u,v,w,p)) \leq \varphi(max\{d(x,u), d(y,v), d(z,w), d(s,p)\}) \qquad (2.5.1)$$

for any $x, y, z, s, u, v, w, p \in X$ for which $x \preceq u$, $y \pm v$, $z \preceq w$ and $s \pm p$.

Suppose either

(a) G is continuous, or

(b) G has the following property:

(1) If non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ $\forall n.$

(2) If non-increasing sequence $y_n \to y$, then $y_n \pm y$ $\forall n.$

If there exists $x_0, y_0, z_0, s_0 \in X$ such that $x_0 \preceq G(x_0, y_0, z_0, s_0), y_0 \pm G(y_0, z_0, s_0, x_0)$ $z_0 \preceq G(z_0, s_0, x_0, y_0)$ and $s_0 \pm G(s_0, x_0, y_0, z_0)$, then there exist $x, y, z, s \in X$ such that $G(x, y, z, s) = x$, $G(y, z, s, x) = y$, $G(z, s, x, y) = z$ and $G(s, x, y, z) = s$. That is, G has a quartet fixed point.

Suppose that for all (x, y, z, s), $(u, v, w, p) \in X \times X \times X \times X$, there exists $(a, b, c, d) \in X \times X \times X \times X$ such that $(G(a,b,c,d), G(b,c,d,a), G(c,d,a,b), G(d,a,b,c))$ is comparable to $(G(x,y,z,s), G(y,z,s,x), G(z,s,x,y), G(s,x,y,z))$ and $(G(u,v,w,p), G(v,w,p,u), G(w,p,u,v), G(p,u,v,w))$.

Then G has a unique quartet fixed point (x, y, z, s).

Proof: Let $x_0, y_0, z_0, s_0 \in X$ be such that $x_0 \preceq G(x_0, y_0, z_0, s_0)$, $y_0 \pm G(y_0, z_0, s_0, x_0)$, $z_0 \preceq G(z_0, s_0, x_0, y_0)$ and $s_0 \preceq G(s_0, x_0, y_0, z_0)$.

Set $x_1 = G(x_0, y_0, z_0, s_0)$, $y_1 = G(y_0, z_0, s_0, x_0)$, $z_1 = G(z_0, s_0, x_0, y_0)$ and

$$s_1 = G(s_0, x_0, y_0, z_0). \qquad (2.5.2)$$

Continuing this process, we can construct sequences $\{x_n\}, \{y_n\}$, $\{z_n\}$ and $\{s_n\}$ in X such that $x_{n+1} = G(x_n, y_n, z_n, s_n)$, $y_{n+1} = G(y_n, z_n, s_n, x_n)$, $z_{n+1} = G(z_n, s_n, x_n, y_n)$ and

$$s_{n+1} = G(s_n, x_n, y_n, z_n). \qquad (2.5.3)$$

Since G has the mixed monotone property, then using mathematical induction,

we have $x_n \preceq x_{n+1}$, $y_{n+1} \pm y_n$, $z_n \preceq z_{n+1}$ and $s_{n+1} \pm s_n$. (2.5.4)

Assume for some $n \in \mathbf{N}$,

$x_n = x_{n+1}$, $y_{n+1} = y_n$, $z_n = z_{n+1}$ and $s_n = s_{n+1}$.

i.e., $(x_n, y_n, z_n, s_n) = (x_{n+1}, y_{n+1}, z_{n+1}, s_{n+1})$

Then by (2.5.3), (x_n, y_n, z_n, s_n) is a quartet fixed point of G.

From now on, assume that for any $n \in \mathbf{N}$, $(x_n, y_n, z_n, s_n) \neq (x_{n+1}, y_{n+1}, z_{n+1}, s_{n+1})$

i.e., $x_n \neq x_{n+1}$ or $y_n \neq y_{n+1}$ or $z_n \neq z_{n+1}$ or $s_n \neq s_{n+1}$. (2.5.5)

Then by (2.2.5), for any $n \in \mathbf{N}$, write

$a_{n+1} = max\{d(x_n, x_{n+1}), d(y_n, y_{n+1}), d(z_n, z_{n+1}), d(s_n, s_{n+1})\} > 1.$

From (2.5.1) and (2.5.3), we have

$$d(x_n, x_{n+1}) = d(G(x_{n-1}, y_{n-1}, z_{n-1}, s_{n-1}), G(x_n, y_n, z_n, z_n)) \le \varphi(max\{d(x_{n-1}, x_n), d(y_{n-1}, y_n), d(z_{n-1}, z_n), d(s_{n-1}, s_n)\}) \quad (2.5.6)$$

$$d(y_{n+1}, y_n) = d(G(y_n, z_n, s_n, x_n), G(y_{n-1}, z_{n-1}, s_{n-1}, x_{n-1})) \le \varphi(max\{d(y_n, y_{n-1}), d(z_n, z_{n-1}), d(s_n, s_{n-1}), d(x_n, x_{n-1})\}) \quad (2.5.7)$$

$$d(z_n, z_{n+1}) = d(G(z_{n-1}, s_{n-1}, x_{n-1}, y_{n-1}), G(z_n, s_n, x_n, y_n),) \le \varphi(max\{d(z_{n-1}, z_n), d(s_{n-1}, s_n), d(x_{n-1}, x_n), d(y_{n-1}, y_n)\}) \quad (2.5.8)$$

and $d(s_{n+1}, s_n) = d(G(s_n, x_n, y_n, z_n, z_n), G(s_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}, s_{n-1}))$

$$\le \varphi(max\{d(s_n, s_{n-1}), d(x_n, x_{n-1}), d(y_n, y_{n-1}), d(z_n, z_{n-1})\}) \quad (2.5.9)$$

Since, $\varphi(t) < t \ \forall t > 1$, from (2.5.6)-(2.5.9) we get that

$$1 < a_{n+1} = max\{d(x_n, x_{n+1}), d(y_n, y_{n+1}), d(z_n, z_{n+1}), d(s_n, s_{n+1})\} \le \varphi(max\{d(x_n, x_{n-1}), d(y_n, y_{n-1}), d(z_n, z_{n-1}), d(s_n, s_{n-1})\}) < max\{d(x_n, x_{n-1}), d(y_n, y_{n-1}), d(z_n, z_{n-1}), d(s_n, s_{n-1})\} = a_n \quad (2.5.10)$$

It follows that $a_{n+1} < a_n$

Thus $\{a_{n+1}\}$ is a positive decreasing sequence. Hence there exists $r \ge 1$ such that a_{n+1} decreases to r.

Suppose $r > 1$, in (2.5.10), we get that $1 < a_{n+1} \le \varphi(a_n) < a_n$. (2.5.11)

On letting $n \to \infty$

$$1 < r \le \lim_{n\to\infty} \varphi(a_n) = \lim_{a_n \to r^+} \varphi(a_n) < r = \lim_{n\to\infty} a_n.$$

which is a contradiction. $\therefore r = 1.$

i.e., $\lim_{n\to\infty}(a_{n+1}) = \lim_{n\to\infty} max\{d(x_n, x_{n+1}), d(y_n, y_{n+1}), d(z_n, z_{n+1}), d(s_n, s_{n+1})\} = 1.$ (2.5.12)

Now we prove that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{s_n\}$ are Cauchy sequences.

Assume the contrary, i.e., $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ or $\{s_n\}$ is not Cauchy sequence.

that is $\lim_{n,m\to\infty} d(x_m, x_n) \ne 1$ or $\lim_{n,m\to\infty} d(y_m, y_n) \ne 1$ or $\lim_{n,m\to\infty} d(z_m, z_n) \ne 1$ or $\lim_{n,m\to\infty} d(s_m, s_n) \ne 1$

This means that there exists $\varepsilon > 1$ for which we can find subsequences of integers $\{m_k\}$ and $\{n_k\}$ with $\{n_k\}$ $>$ $\{m_k\} > k$

such that $max\{d(x_{m_k}, x_{n_k}), d(y_{m_k}, y_{n_k}), d(z_{m_k}, z_{n_k}), d(z_{m_k}, z_{n_k})\} \ge \varepsilon$ (2.5.13)

Further, corresponding to $\{m_k\}$ we can choose $\{n_k\}$ in such a way that it is the smallest integer with $\{n_k\} > \{m_k\}$ and satisfying (2.5.13)

Then $max\{d(x_{m_k}, x_{n_{k-1}}), d(y_{m_k}, y_{n_{k-1}}), d(z_{m_k}, z_{n_{k-1}}), d(s_{m_k}, s_{n_{k-1}})\} \le \varepsilon$ (2.5.14)

by triangular inequality and (2.2.14), we have

$$d(x_{m_k}, x_{n_k}) \le d(x_{m_k}, x_{n_{k-1}}).\{d(x_{n_{k-1}}, x_{n_k}) < \varepsilon.\{d(x_{n_{k-1}}, x_{n_k})$$

Thus, by (2.5.12), we get that

$$\varepsilon \le \lim_{k\to\infty} d(x_{m_k}, x_{n_k}) \le \lim_{k\to\infty} d(x_{m_k}, x_{n_{k-1}}) \le \varepsilon \quad (2.5.16)$$

Similarly, we have $\varepsilon \le \lim_{k\to\infty} d(y_{m_k}, y_{n_k}) \le \lim_{k\to\infty} d(y_{m_k}, y_{n_{k-1}}) \le \varepsilon$ (2.5.16)

$$\varepsilon \le \lim_{k\to\infty} d(z_{m_k}, z_{n_k}) \le \lim_{k\to\infty} d(z_{m_k}, z_{n_{k-1}}) \le \varepsilon \quad (2.5.17)$$

and $\varepsilon \le \lim_{k\to\infty} d(s_{m_k}, s_{n_k}) \le \lim_{k\to\infty} d(s_{m_k}, s_{n_{k-1}}) \le \varepsilon$ (2.5.18)

Again by (2.5.14), we have

$$d(x_{m_k}, x_{n_k}) \le d(x_{m_k}, x_{m_{k-1}}).d(x_{m_{k-1}}, x_{n_{k-1}}).d(x_{n_{k-1}}, x_{n_k}) \le d(x_{m_k}, x_{m_{k-1}}).d(x_{m_{k-1}}, x_{m_k}).d(x_{m_k}, x_{n_{k-1}}).d(x_{n_{k-1}}, x_{n_k})$$

Letting $k \to \infty$ and using (2.5.12), we get

$$\varepsilon \le \lim_{k\to\infty} d(x_{m_k}, x_{n_k}) \le 1.1.\varepsilon.1. = \varepsilon. \qquad (2.5.19)$$

Similarly,

$$\varepsilon \le \lim_{k\to\infty} d(y_{m_k}, y_{n_k}) \le \varepsilon. \qquad (2.5.20)$$

$$\varepsilon \le \lim_{k\to\infty} d(z_{m_k}, z_{n_k}) \le \varepsilon. \qquad (2.5.21)$$

and $\varepsilon \le \lim_{k\to\infty} d(s_{m_k}, s_{n_k}) \le \varepsilon.$ (2.5.22)

Now, using (2.5.13) and (2.5.19)-(2.5.22), we have

$$\lim_{k\to\infty}\{d(x_{m_k}, x_{n_k}), d(y_{m_k}, y_{n_k}), d(z_{m_k}, z_{n_k}), d(s_{m_k}, s_{n_k})\} = \varepsilon. \qquad (2.5.23)$$

Now using (2.5.1), we obtain that

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &= d(G(x_{m_{k-1}}, y_{m_{k-1}}, z_{m_{k-1}}, s_{m_{k-1}}), G(x_{n_{k-1}}, y_{n_{k-1}}, z_{n_{k-1}}, s_{n_{k-1}})) \\ &\le \varphi(max\{d(x_{m_{k-1}}, x_{n_{k-1}}), d(y_{m_{k-1}}, y_{n_{k-1}}), d(z_{m_{k-1}}, z_{n_{k-1}}), d(s_{m_{k-1}}, s_{n_{k-1}})\}) \qquad (2.5.24) \end{aligned}$$

$$\begin{aligned} d(y_{m_k}, y_{n_k}) &= d(G(y_{m_{k-1}}, z_{m_{k-1}}, s_{m_{k-1}}, x_{m_{k-1}}), G(y_{n_{k-1}}, z_{n_{k-1}}, s_{n_{k-1}}, x_{n_{k-1}})) \\ &\le \varphi(max\{d(y_{m_{k-1}}, y_{n_{k-1}}), d(z_{m_{k-1}}, z_{n_{k-1}}), d(s_{m_{k-1}}, s_{n_{k-1}}), d(x_{m_{k-1}}, x_{n_{k-1}})\}) \qquad (2.5.25) \end{aligned}$$

$$\begin{aligned} d(z_{m_k}, z_{n_k}) &= d(G(z_{m_{k-1}}, s_{m_{k-1}}, x_{m_{k-1}}, y_{m_{k-1}}), G(z_{n_{k-1}}, s_{n_{k-1}}, x_{n_{k-1}}, y_{n_{k-1}})) \\ &\le \varphi(max\{d(z_{m_{k-1}}, z_{n_{k-1}}), d(s_{m_{k-1}}, s_{n_{k-1}}), d(x_{m_{k-1}}, x_{n_{k-1}}), d(y_{m_{k-1}}, y_{n_{k-1}})\}) \qquad (2.5.26) \end{aligned}$$

$$\begin{aligned} d(s_{m_k}, s_{n_k}) &= d(G(s_{m_{k-1}}, x_{m_{k-1}}, y_{m_{k-1}}, z_{m_{k-1}}), G(s_{n_{k-1}}, x_{n_{k-1}}, y_{n_{k-1}}, z_{n_{k-1}})) \\ &\le \varphi(max\{d(s_{m_{k-1}}, s_{n_{k-1}}), d(x_{m_{k-1}}, x_{n_{k-1}}), d(y_{m_{k-1}}, y_{n_{k-1}}), d(z_{m_{k-1}}, z_{n_{k-1}})\}) \qquad (2.5.27) \end{aligned}$$

From (2.5.24) to (2.5.27) we get that

$$\begin{aligned} &max\{d(x_{m_k}, x_{n_k}), d(y_{m_k}, y_{n_k}), d(z_{m_k}, z_{n_k}), d(s_{m_k}, s_{n_k})\} \\ &\quad \le \phi(max\{d(x_{m_{k-1}}, x_{n_{k-1}}), d(y_{m_{k-1}}, y_{n_{k-1}}), d(z_{m_{k-1}}, z_{n_{k-1}}), d(s_{m_{k-1}}, s_{n_{k-1}})\}) = \varepsilon \qquad (2.5.28) \end{aligned}$$

Letting $k \to \infty$ in (2.5.28) and having in mind (2.5.23), we get

$1 < \varepsilon \le \lim_{t\to \grave{o}^+} \varphi(t) < \varepsilon$, which is a contradiction.

Thus $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{s_n\}$ are cauchy sequences in X. Since X is a complete multiplicative metric space, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{s_n\}$ are convergent sequences.

So there exist $x, y, z, s \in X$ such that $x_n \to x$, $y_n \to y$, $z_n \to z$ and $s_n \to s$ (2.5.29)

Suppose now the assumption (a) holds, that is G is continuous.

By (2.5.3) and (2.5.29) we obtain that

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} G(x_n, y_n, z_n, s_n) = G(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n, \lim_{n\to\infty} z_n, \lim_{n\to\infty} s_n) = G(x, y, z, s)$$

Similarly $y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} G(y_n, z_n, s_n, x_n) = G(\lim_{n\to\infty} y_n, \lim_{n\to\infty} z_n, \lim_{n\to\infty} s_n, \lim_{n\to\infty} x_n) = G(y, z, s, x)$

$$z = \lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} G(z_n, s_n, x_n, y_n) = G(\lim_{n\to\infty} z_n, \lim_{n\to\infty} s_n, \lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = G(z, s, x, y)$$

and $s = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} G(s_n, x_n, y_n, z_n) = G(\lim_{n\to\infty} s_n, \lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n, \lim_{n\to\infty} z_n) = G(s, x, y, z)$

$\therefore (x, y, z, s)$ is a quartet fixed point of G.

Suppose the assumption (b) holds,

i.e., $\{x_n\}$, $\{z_n\}$ are non-decreasing with $x_n \to x$, $z_n \to z$ and also $\{y_n\}$, $\{s_n\}$ are non-increasing with $y_n \to y$, $s_n \to s$, then we have $x_n \preceq x$, $z_n \preceq z$, $y_n \pm y$, and $s_n \pm s$.

for all n, Now consider

$$\begin{aligned} d(x, G(x, y, z, s)) &\le d(x, x_{n+1}).d(x_{n+1}, G(x, y, z, s)) \\ &= d(x, x_{n+1}).d(G(x_n, y_n, z_n, s_n), G(x, y, z, s)) \\ &\le d(x, x_{n+1}).\varphi(max\{d(x_n, x), d(y_n, y), d(z_n, z), d(s_n, s)\}) \qquad (2.5.30) \end{aligned}$$

For infinitely many n, $\varphi(a_n) < a_n$

$$1 \le \lim_{n\to\infty} d(x, G(x, y, z, s)) \le \lim_{n\to\infty}[d(x, x_{n+1}).max\{d(x_n, x), d(y_n, y), d(z_n, z), d(s_n, s)\}] \to 1$$

$1 \le \lim_{n\to\infty} d(x, G(x,y,z,s)) \le 1$

$d(x, G(x,y,z,s)) = 1$.

$\therefore x = G(x,y,z,s)$.

Similarly we can show that $y = G(y,z,s,x)$, $z = G(z,s,y,x)$ and $s = G(s,x,y,z)$

Hence (x,y,z,s) is a quartet fixed point of G .

Uniqueness of quartet fixed point:

Let (x,y,z,s) and (u,v,w,p) be two quartet fixed points of G .

Suppose $(x,y,z,s) \preceq (u,v,w,p)$ i.e., $x \preceq u$, $y \pm v$, $z \preceq w$ and $s \pm p$.

$d(x,u) = d(G(x,y,z,s), G(u,v,w,p)) \le \varphi(max\{d(x,u), d(y,v), d(z,w), d(s,p)\})$

$= \varphi(d(x,u) < d(x,u)$ $\quad (if \; d(x,u) \, is \max imum)$ $(d(x,u) < d(x,u)$

, a contradiction, if $d(x,u) > 1$.

$\therefore d(x,u) = 1, d(y,v) = 1, d(z,w) = 1, d(s,p) = 1$

$\therefore x = u, y = v, z = w$ and $s = p$

$\therefore (x,y,z,s) = (u,v,w,p)$.

By our assumption, there exists $(a,b,c,d) \in X \times X \times X \times X$ such that

$(G(a,b,c,d), G(b,c,d,a), G(c,d,a,b), G(d,a,b,c))$ is comparable to

$(G(x,y,z,s), G(y,z,s,x), G(z,s,x,y), G(s,x,y,z))$ and $(G(u,v,w,p), G(v,w,p,u), G(w,p,u,v), G(p,u,v,w))$.

Define sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ such that $a_0 = a$, $b_0 = b$, $c_0 = c$ and $d_0 = d$

and for any $n \ge 1$,

$a_n = G(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1})$, $b_n = G(b_{n-1}, c_{n-1}, d_{n-1}, a_{n-1})$,

$c_n = G(c_{n-1}, d_{n-1}, a_{n-1}, b_{n-1})$ and $d_n = G(d_{n-1}, a_{n-1}, b_{n-1}, c_{n-1}) \quad \forall n$ (2.5.31)

for all n , further set $x_0 = x$, $y_0 = y$, $z_0 = z$, $s_0 = s$ and $u_0 = u$, $v_0 = v$, $w_0 = w$, $p_0 = p$

and in the same way, we define the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{s_n\}$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{p_n\}$.

Then it is easy to show that $G(x,y,z,s) = x_n$, $G(y,z,s,x) = y_n$, $G(z,s,x,y) = z_n$ and $G(s,x,y,z) = s_n$.

also $G(u,v,w,p) = u_n$, $G(v,w,p,u) = v_n$, $G(w,p,u,v) = w_n$ and $G(p,u,v,w) = p_n \quad \forall n \ge 1$. (2.5.32)

since $(G(x,y,z,s), G(y,z,s,x), G(z,s,x,y), G(s,x,y,z))$ = $(x_1, y_1, z_1, s_1) = (x,y,z,s)$ is comparable to

$(G(a,b,c,d), G(b,c,d,a), G(c,d,a,b), G(d,a,b,c))$ = (a_1, b_1, c_1, d_1)

Then it is enough to show that $(x,y,z,s) \pm (a_1, b_1, c_1, d_1)$

Recursively, we get that $(x,y,z,s) \pm (a_n, b_n, c_n, d_n) \quad \forall n$ (2.5.33)

By (2.5.33) and (2.5.1), we have

$d(x, a_{n+1}) = d(G(x,y,z,s), G(a_n, b_n, c_n, d_n))$

$\le \varphi(max\{d(x,a_n), d(y,b_n), d(z,c_n), d(s,d_n)\})$ (2.5.34)

$d(b_{n+1}, y) = d(G(b_n, c_n, d_n, a_n), G(y,z,s,x))$

$\le \varphi(max\{d(b_n,y), d(c_n,z), d(d_n,s), d(a_n,x)\})$ (2.5.35)

$d(z, c_{n+1}) = d(G(z,s,x,y), G(c_n, d_n, a_n, b_n))$

$\le \varphi(max\{d(z,c_n), d(s,d_n), d(x,a_n), d(y,b_n)\})$ (2.5.36)

and $d(d_{n+1}, s) = d(G(d_n, a_n, b_n, c_n), G(s,x,y,z))$

$\le \varphi(max\{d(d_n,s), d(a_n,x), d(b_n,y), d(c_n,z)\})$ (2.5.37)

It follows from (2.5.34)-(2.5.37) that

$max\{d(x,a_{n+1}), d(y,b_{n+1}), d(z,c_{n+1}), d(s,d_{n+1})\} \le \varphi(max\{d(x,a_n), d(y,b_n), d(z,c_n), d(s,d_n)\})$

there fore, for each $n \ge 1$,

$(max\{d(x,a_n), d(y,b_n), d(z,c_n), d(s,d_n)\}) \le \varphi(max\{d(x,a_{n-1}), d(y,b_{n-1}), d(z,c_{n-1}, d(s,d_{n-1}))\})$

$$\leq \varphi^n \ (max\{d(x,a_0),d(y,b_0),d(z,c_0),d(s,d_0)\}) \quad (2.5.38)$$

Since $\varphi(t) < t$ and $\lim_{r\to t^+}\varphi(r) < t \Rightarrow \lim_{n\to\infty}\varphi^n(t) = 1$

for each $t > 1$, thus, from(2.5.38)

$\lim_{n\to\infty} \max\{d(x,a_n),d(y,b_n),d(z,c_n),d(s,d_n)\} = 1$

i.e., $\lim_{n\to\infty} d(x,a_n) = 1$, $\lim_{n\to\infty} d(y,b_n) = 1$,

$\lim_{n\to\infty} d(z,c_n) = 1$ and $\lim_{n\to\infty} d(s,d_n) = 1$. (2.5.39)

Analogously, we show that

$\lim_{n\to\infty} d(u,a_n) = 1$, $\lim_{n\to\infty} d(v,b_n) = 1$,

$\lim_{n\to\infty} d(w,c_n) = 1$, and $\lim_{n\to\infty} d(p,d_n) = 1$ (2.5.40)

Combining (2.5.39) and (2.5.40), it follows that (x, y, z, s) and (u, v, w, p) are equal.

$\therefore x = u$, $y = v$, $z = w$, and $s = p$.

Hence Uniqueness holds.

Now we give two examples in support of our Theorem 2.5. It may be observed that a cursory look at the examples will not yield the quartet fixed point. We conclude the existence of the quartet fixed point only via our Theorem 2.5.

Example 2.6. Let $X = [\frac{1}{2}, 64]$ with the multiplicative metric $d(x, y) = a^{|x-y|}$, $a > 1$ for all $x, y \in X$ and the usual ordering $\preceq$. Clearly, (X, d) is a complete multiplicative metric space.

Let $G: X\times X\times X\times X$ be defined by

$$G(x,y,z,s) = \log 8(\frac{\sqrt{xz}}{ys})^{\frac{1}{12}} + 1 \quad \forall \ x,y,z,s \in X \text{ and } \varphi(t) = t^{\frac{2}{3}},\ t\in[1,\infty).$$

Here G has the mixed monotone property and is continuous.

Taking $x,y,z,s,u,v,w,p \in X$ for which $x \preceq u$, $y \pm v$, $z \preceq w$ and $s \pm p$,

we have

$$\begin{aligned}
d(G(x,y,z,s),G(u,v,r,t)) &= a^{|G(x,y,z,s)-G(u,v,w,p)|}\\
&= a^{|(\log 8(\frac{\sqrt{xz}}{ys})^{12}+1)-(\log 8(\frac{\sqrt{uw}}{vp})^{12}+1)|}\\
&= a^{|\{\log 8+\frac{1}{24}(\log x+\log z-\frac{1}{12}(\log y+\log s)+1\}-\{\log 8+\frac{1}{24}(\log u+\log w-\frac{1}{12}(\log v+\log p)+1\}|}\\
&\leq a^{\frac{1}{24}\{|\log x-\log u|+|\log z-\log w|+2|\log v-\log y|+2|\log p-\log s\}|}\\
&\leq a^{\frac{1}{12}\{|\log x-\log u|+|\log z-\log w|+|\log v-\log y|+|\log p-\log s\}|}\\
&= (a^{|\log x-\log u|})^{\frac{1}{12}}).(a^{|\log z-\log w|})^{\frac{1}{12}}) . (a^{|\log y-\log v|})^{\frac{1}{12}} . (a^{|\log s-\log p|})^{\frac{1}{12}})\\
&= d(\log x,\log u)^{\frac{1}{12}}).d(\log z,\log w)^{\frac{1}{12}}) . d(\log y,\log v)^{\frac{1}{12}} . d(\log s,\log p)^{\frac{1}{12}})\\
&\leq \{d(x,u)^{\frac{1}{6}}) . d(z,w)^{\frac{1}{6}}) . d(y,v)^{\frac{1}{6}}) . d(s,p)^{\frac{1}{6}}\}\\
&\leq max\{d(x,u)^{\frac{2}{3}}) . d(z,w)^{\frac{2}{3}}) . d(y,v)^{\frac{2}{3}}) . d(s,p)^{\frac{2}{3}}\}\\
&= max\{\varphi(d(x,u)),\varphi(d(z,w)),\varphi(d(y,v)),\varphi(d(s,p))\}\\
&= \varphi(max\{d(x,u),d(z,w),d(y,v),d(s,p)\}
\end{aligned}$$

which is a contractive condition of (2.5.1). Moreover, taking $x_0 = z_0 = 1$ and $y_0 = s_0 = 64$,

we have $x_0 \preceq G(x_0, y_0, z_0, s_0)$, $y_0 \pm G(y_0, z_0, s_0, x_0)$,

$z_0 \preceq G(z_0, s_0, x_0, y_0)$ and $s_0 \pm G(s_0, x_0, y_0, z_0)$.

Therefore all the conditions of theorem 2.5 hold.

Suppose $x = y = z = s$. Then $G(x, y, z, s) = \log 8(\frac{\sqrt{xz}}{ys})^{\frac{1}{12}} + 1$

Now we show that there exists $x \in (1, 64)$ such that $G(x, x, x, x) = x$,

i.e., x is a quartet fixed point of G.

Consider $G(x, x, x, x) = \log 8(\frac{1}{x})^{\frac{1}{12}} + \log e$

Now $G(x, x, x, x) = x \quad \Rightarrow \log[e.8(\frac{1}{x})^{\frac{1}{12}}] = x = \log y$ (say)

$$\Rightarrow e.8(\frac{1}{x})^{\frac{1}{12}} = y = e^x$$

$$\Rightarrow \frac{8}{x^{\frac{1}{12}}} = e^{x-1}$$

$$\Rightarrow 8 = e^{x-1}.x^{\frac{1}{12}}$$

$$\Rightarrow 8 - e^{x-1}.x^{\frac{1}{12}} = 0.$$

Write $f(t) = 8 - e^{t-1}.t^{\frac{1}{12}}$ in [1,64]

Then $f(1) = 7 > 0$ and $f(64) = 8 - e^{63}.2^{\frac{1}{2}} < 0$

$\therefore\ \exists\ x \in (1, 64)$ such that $f(x) = 0$ (by mean value theorem)

i.e., $8 - e^{x-1}.x^{\frac{1}{12}} = 0$. $\Rightarrow 8 = e^{x-1}.x^{\frac{1}{12}}$.

$$\log(e.8(\frac{1}{x})^{\frac{1}{12}} = x.$$

i.e., $\log(8(\frac{1}{x})^{\frac{1}{12}} + 1 = x$.

i.e., $G(x, x, x, x) = x$

therefore x is a quartet fixed point of G.

The exact value of the quartet fixed point is not known but existence is guaranteed.

Example 2.7. Let $X = [\frac{1}{2}, 64]$ with the multiplicative metric $d(x, y) = a^{|x-y|}$, $a > 1$ for all $x, y \in X$ and the usual ordering $\preceq$. Clearly, (X, d) is a complete multiplicative metric space.

Let $G : X \times X \times X \times X$ be defined by

$G(x, y, z, s) = \log 8(\sqrt{\frac{xz}{ys}})^{\frac{1}{12}} + 1 \ \forall\ x, y, z, s \in X$ and $\phi(t) = t^{\frac{2}{3}}$, $t \in [1, \infty)$.\\

Here G has the mixed monotone property and is continuous.

Taking $x, y, z, s, u, v, w, p \in X$ for which $x \preceq u$, $y \pm v$, $z \preceq w$ and $s \pm p$,

we have

$$d(G(x, y, z, s), G(u, v, r, t))) = a^{|G(x,y,z,s) - G(u,v,w,p)|}$$

$$= a^{|(\log 8(\sqrt{\frac{xz}{ys}})^{12} + 1) - (\log 8(\sqrt{\frac{uw}{vp}})^{12} + 1)|}$$

$$= a^{|\{\log 8 + \frac{1}{24}(\log x + \log z - \frac{1}{24}(\log y + \log s) + 1\} - \{\log 8 + \frac{1}{24}(\log u + \log w - \frac{1}{24}(\log v + \log p) + 1\}|}$$

$$\leq a^{\frac{1}{24}\{|\log x - \log u| + |\log z - \log w| + |\log v - \log y| + |\log p - \log s\}|}$$

$$\leq a^{\frac{1}{24}\{|\log x - \log u| + |\log z - \log w| + |\log v - \log y| + |\log p - \log s\}|}$$

$$\leq a^{\frac{1}{12}\{|\log x-\log u|+|\log z-\log w|+|\log v-\log y|+|\log p-\log s\}|}$$

$$= (a^{|\log x-\log u|})^{\frac{1}{12}}).(a^{|\log z-\log w|})^{\frac{1}{12}}) . (a^{|\log y-\log v|})^{\frac{1}{12}} . (a^{|\log s-\log p|})^{\frac{1}{12}})$$

$$= d(\log x,\log u)^{\frac{1}{12}}).d(\log z,\log w)^{\frac{1}{12}}) . d(\log y,\log v)^{\frac{1}{12}} . d(\log s,\log p)^{\frac{1}{12}})$$

$$\leq \{d(x,u)^{\frac{1}{6}}) . d(z,w)^{\frac{1}{6}}) . d(y,v)^{\frac{1}{6}}) . d(s,p)^{\frac{1}{6}}\}$$

$$\leq max\{d(x,u)^{\frac{2}{3}}) . d(z,w)^{\frac{2}{3}}) . d(y,v)^{\frac{2}{3}}) . d(s,p)^{\frac{2}{3}}\}$$

$$= max\{\varphi(d(x,u)),\varphi(d(z,w)),\varphi(d(y,v)),\varphi(d(s,p))\}$$

$$= \varphi(max\{d(x,u),d(z,w),d(y,v),d(s,p)\}$$

which is a contractive condition of (2.5.1). Moreover, taking $x_0 = z_0 = 1$ and $y_0 = s_0 = 64$,

we have $x_0 \preceq G(x_0, y_0, z_0, s_0)$, $y_0 \pm G(y_0, z_0, s_0, x_0)$,

$z_0 \preceq G(z_0, s_0, x_0, y_0)$ and $s_0 \pm G(s_0, x_0, y_0, z_0)$.

since $G(x,x,x,x) = \log 8(\sqrt{\frac{xx}{xx}})^{\frac{1}{12}} + 1 = \log 8 + 1 = 3\log 2 + 1$.

we observe that $3\log 2 + 1$ is a quartet fixed point of G.

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